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ROOT STRUCTURE OF SUPER HYPERBOLIC KAC-MOODY ALGEBRAS $SH_{71}(3)$

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ABSTRACT

In this Paper, we consider a family of indefinite, non-hyperbolic type called as Super Hyperbolic type. Let $SH_{71}^{(3)}$ be the Super Hyperbolic Dynkin diagrams of rank 4 obtained from the rank 3 hyperbolic diagrams of $H_{71}^{(3)}$. The complete classification of Dynkin diagrams associated with $SH_{71}^{(3)}$ is obtained here. There are 219 non-isomorphic connected Dynkin diagrams associated with the family $SH_{71}^{(3)}$. All the GCM associated with these family are symmetrizable. Properties of roots are also computed

Keywords: *Dynkin diagrams, Kac-Moody algebras, ,Imaginary, Non-isomorphic, Real, Super Hyperbolic s.*

I. INTRODUCTION

Kac Moody algebra (abbreviated as KM-algebras) is the rapidly developing fields of Mathematical research, introduced by V.Kac and R.Moody simultaneously and independently in 1968. Kac Moody algebras have wide application to other topics in Mathematics and Mathematical Physics. Kac Moody algebras are broadly classified into three types: finite, affine and indefinite type. The classification of Dynkin diagrams and Generalized Cartan Matrices (GCM) of finite and affine type and a class of indefinite type namely hyperbolic have already been given. Among the indefinite type, hyperbolic Dynkin diagrams are natural extension of affine type, whose classification is also already known.

II. PRELIMINARIES

The basic definitions and notations are used as in [2] - [6].

Let g(A) be the Kac-Moody algebra associated with a GCM $A=(a_{ij})_{i,j=1}^{n}$ and generated by the elements ei , fi , i=1,2,…n and H with the following relations :

$$[h,h'] = 0, \quad h,h' \in H$$

$$[e_i, f_j] = \delta_{ij}\alpha_i^{\vee}$$

$$[h,e_j] = \alpha_j(h)e_j$$

$$[h, f_j] = -\alpha_j(h) f_j \ , \ i, j \in N$$

$$(ade_i)^{1-a_{ij}} e_j = 0$$

$$(adf_i)^{1-a_{ij}} f_j = 0 \ , \forall \ i \neq j, \ i, j \in N$$

Let $(H,\Pi,\Pi^{\vee})$ be the realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ is a triple, where l is the rank of A, H is a 2n - l dimensional complex vector space. The linearly independent subsets of H* and H are $\Pi=\{\alpha_1,\alpha_2,...,\alpha_n\}$

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and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_n^\vee\}$ respectively, satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis and elements of Π are called simple roots.

Let the root space decomposition of g(A) be given by $g(A) = \underset{\alpha \in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A) / [h, x] = \alpha(h)x, \ for \ \ all \ \ h \in H\}$.

A Dynkin diagram S(A) associated with the GCM A is defined as follows: S(A) has n vertices and vertices i and j are connected by max{|aij|, |aji|} number of lines if aij.aji≤4 and there ia an arrow pointing towards i if |aij|>1. If aij.aji>4, i and j are connected by a bold faced edge, equipped with the ordered pair (|aij|, |aji|) of integers.

We define a GGCM $A = (a_{ij})_{i,j=1}^n$ is of Super Hyperbolic type (abbreviated as SH type) if A is not of hyperbolic type and any indecomposable proper principal submatrix of A is of finite, affine or hyperbolic type.[1]

A root $\alpha \in \Delta$ is called real, if there exists a $w \in W$ such that w(α) is a simple root, and a root which is not real is called an imaginary root. An imaginary root α is called isotropic if (α,α) = 0.

III. COMPLETE CLASSIFICATION OF DYNKIN DIAGRAMS OF SUPER HYPERBOLIC KM-ALGEBRAS $SH_{71}^{(3)}$

Let us consider the Super Hyperbolic KM-algebra $SH_{71}^{(3)}$ whose associated GCM is

$A = \begin{pmatrix} 2 & -2 & -2 & -a \\ -2 & 2 & -2 & -b \\ -2 & -2 & 2 & -c \\ -e & -f & -g & 2 \end{pmatrix}$, a,b,c,e,f,g are non negative integers.

Here A is symmetrizable, with the conditions, bg = cf and ce = ag. i.e. A = DB,

where $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c/g \end{pmatrix}, B = \begin{pmatrix} 2 & -2 & -2 & -a \\ -2 & 2 & -1 & -b \\ -2 & -1 & 2 & -c \\ -a & -b & -c & 2g/c \end{pmatrix}$.

Note: In this paper, the positive integers a,b,c,e,f,g are not all 0 simultaneously.

Theorem 3.1: There are 219 connected non-isomorphic Dynkin diagrams associated with the family $SH_{71}^{(3)}$. There are 219 non-isomorphic connected Dynkin diagrams associated with the family $SH_{71}^{(3)}$.

Proof: Let us consider the Super hyperbolic KM algebra $SH_{71}^{(3)}$, which contains 3 vertices and the associated Dynkin diagram is given in [fig. 1]

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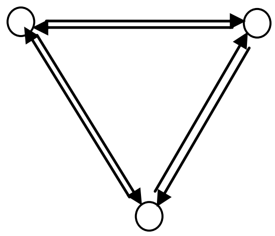


Fig. 1

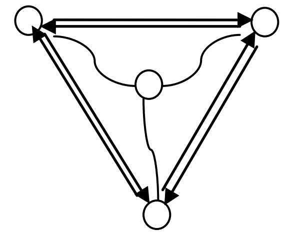
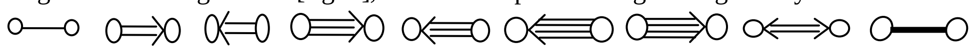


Fig. 2

Now, by adding one more vertex to the hyperbolic KM algebra $H_{71}^{(3)}$ [fig. 1], in which all the other three vertices are joined to the fourth vertex with the 9 possible edges, which is denoted by  and the obtained Super hyperbolic KM algebra $SH_{71}^{(3)}$ is given in [fig. 2], where the 9 possible edges are given by



The complete classification and the the existence of isomorphic and non-isomorphic Dynkin diagrams of Super hyperbolic KM algebras $SH_{71}^{(3)}$ is explained by the following cases .

Case i) Now, the three vertices 1,2 and 3 are connected independently to the fourth vertex, by the 9 possible edges. Here, all the edges connecting the vertices 1, 2 and 3 are all identical and when we connect the 4th vertex with the other 3 vertices independently, there exists 18 isomorphic Dynkin diagrams among the 27 Dynkin diagrams. Therefore, we consider only Fig.3 and we get 9 possible number of connected non-isomorphic Super hyperbolic Dynkin diagrams.

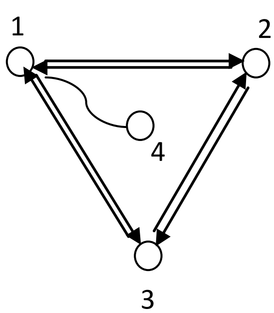


Fig. 3

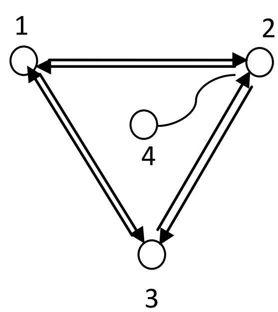


Fig. 4

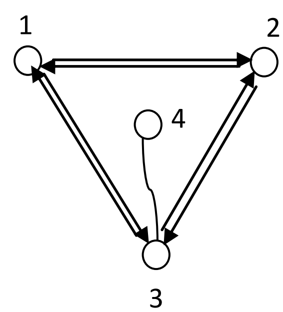


Fig.5

Case ii) In this case, we connect the the fourth vertex to any of the two vertices by the 9 possible edges. Therefore, we get 3 x 81 = 243 Dynkin diagrams, in which 198 Dynkin diagrams are isomorphic and finally we get, 45 connected non-isomorphic Dynkin diagrams by considering fig.6 alone.

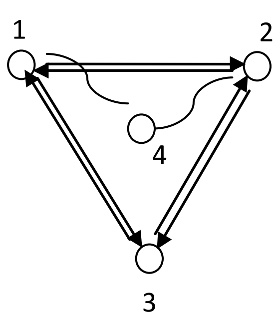


Fig. 6

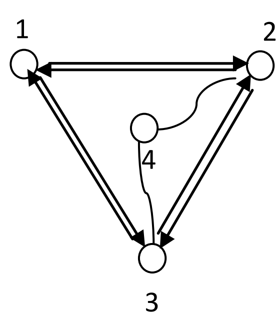


Fig. 7

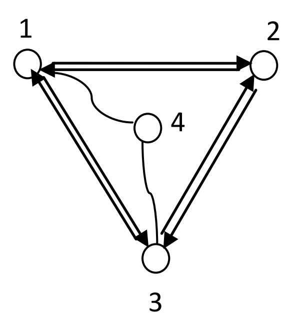


Fig.8

Case iii) In this case, we connect the fourth vertex to all the vertices 1, 2 and 3 simultaneously, we get 749 Dynkin diagrams, in which 584 are isomorphic. By excluding these diagrams, we get 165 non-isomorphic Dynkin diagrams.

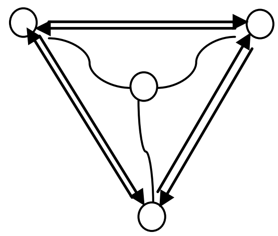


Fig.9

Therefore, totally we get, 9 + 45 + 165 = 219 connected non-isomorphic Dynkin diagrams of Super hyperbolic KM-algebras $SH_{71}^{(3)}$.

IV. ROOT SYSTEM OF SUPER HYPERBOLIC KM-ALGEBRAS $SH_{71}^{(3)}$

In this section, we explain the root properties of $SH_{71}^{(3)}$, using an example.

Example 3.1: Let us consider the GCM of super hyperbolic KM-algebras $A = \begin{pmatrix} 2 & -2 & -2 & -1 \\ -2 & 2 & -2 & -1 \\ -2 & -2 & 2 & -c \\ -1 & -1 & -g & 2 \end{pmatrix}$, which is symmetrizable. Therefore, A = DB = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c/g \end{pmatrix} \begin{pmatrix} 2 & -2 & -2 & -1 \\ -2 & 2 & -2 & -1 \\ -2 & -2 & 2 & -c \\ -1 & -1 & -c & 2g/c \end{pmatrix}$

The non-degenerate symmetric bilinear form (,), is given by

$(\alpha_1,\alpha_1)= (\alpha_2,\alpha_2)= (\alpha_3,\alpha_3)=2$;

$(\alpha_2,\alpha_3)=(\alpha_1,\alpha_3)= (\alpha_1,\alpha_2)=((\alpha_2,\alpha_1) = (\alpha_3,\alpha_2)=(\alpha_3,\alpha_1)= = -2$,

$(\alpha_4,\alpha_1) = (\alpha_1,\alpha_4) = (\alpha_2,\alpha_4)= (\alpha_4,\alpha_2)= -1$; $(\alpha_3,\alpha_4)= (\alpha_4,\alpha_3)= -c$; $(\alpha_4,\alpha_4)=2g/c$.

All fundamental roots are real.

Short and Long Real Roots:

If g = c, then all the roots are equal having the length 2.

If lg> c, then α_1, α_2, α_3 is the short root and $\alpha 4$ is the long root.

If 2g = c, then $\alpha 4$ is the long root and $l\neq 1$ and 2.

Roots of Height 2:

$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) =0$, $\alpha_1+\alpha_2$ is an isotropic root.

$(\alpha_1+\alpha_4, \alpha_1+\alpha_4) = 2g/c$, is a real root.

$(\alpha_3+\alpha_4, \alpha_3+\alpha_4) = 2+2g/c-2c$.

$\alpha_3+\alpha_4$ is a real root if g > c and g/c > c;

$\alpha_3+\alpha_4$ is a imaginary root if g > c and g/c < c.

Roots of Height 3:

$(\alpha_1+2\alpha_2, \alpha_1+2\alpha_2) =2$, $\alpha_1+2\alpha_2$ is real root.

$(2\alpha_1+\alpha_4, 2\alpha_1+\alpha_4) = 4 + 2g/c$ which is a real root.

$(\alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3)= 0$, $\alpha_1+\alpha_2+\alpha_3$ is an isotropic root.

$(\alpha_1+2\alpha_4, \alpha_1+2\alpha_4) = -2+8g/c$;
$\alpha_1+2\alpha_4$ is real if $8g/c > 2$ and imaginary if $8g/c < 2$.
$\alpha_1+2\alpha_4$ is isotropic if $8g/c = 2$.

$(2\alpha_3+\alpha_4, 2\alpha_3+\alpha_4) = 8+2g/c-4c$.
$2\alpha_3+\alpha_4$ is a real root if $g/c < 3$;
$2\alpha_3+\alpha_4$ is a imaginary root if $g/c \geq 3$.

Roots of Height 4:
$(3\alpha_1+\alpha_2, 3\alpha_1+\alpha_2) = 8$, $3\alpha_1+\alpha_2$ is a real root.
$(2\alpha_2+2\alpha_4, 2\alpha_2+2\alpha_4) = 8g/c$ is a real root.

$(2\alpha_1+2\alpha_2, 2\alpha_1+2\alpha_2) =0$, $2\alpha_1+2\alpha_2$ is an isotropic root.

$(2\alpha_1+\alpha_2+\alpha_3, 2\alpha_1+\alpha_2+\alpha_3)= -8$, $2\alpha_1+\alpha_2+\alpha_3$ is an imaginary root.

$(\alpha_1+2\alpha_3+\alpha_4, \alpha_1+2\alpha_3+\alpha_4)= -2g/c-4c$;
$\alpha_1+2\alpha_3+\alpha_4$ is a real root if $g/c > c$ and $g/c > 2$;
$\alpha_1+2\alpha_3+\alpha_4$ is an imaginary root if $g/c < c$.

V. CONCLUSION

Thus, we have delimited the complete classification of Dynkin diagrams for the super hyperbolic KM-algebra $SH_{71}^{(3)}$ and proved the root properties with an example. Similarly, we can explicit the result for other families of super hyperbolic type and we can extend it to the Generalized Kac-Moody algebras of Super Hyperbolic type.

REFERENCES

1. *CHEN Hongji and LIU Bin, "Super Hyperbolic Type Kac-Moody Lie Algebra", System Science and Mathematical sciences, Vol.10 No.4, 329-332, 1997.*
2. *K.Erdmann, Mark J.Wildon, "Introduction to Lie Algebras", Springer London, 2011.*
3. *Kac, V.G. (1990), "Infinite Dimensional Lie Algebra", 3rd ed. Cambridge: Cambridge University Press. (1990)*
4. *R.V. Moody, A new class of Lie algebras, J. Algebra, 10, (1968), 211-230.*
5. *Neelacanta Stanumoorthy, "Introduction to Finite and Infinite Dimensional Lie (Super) algebras", Academic Press, April 2016.*
6. *Wan Zhe-Xian, "Introduction to Kac-Moody Algebra", Singapore: World Scientific Publishing Co.Pvt.Ltd, (1991).*